



On High-Resolution Entropy-Consistent Flux with Slope Limiter for Hyperbolic Conservation Laws

Xuan Ren^{1,2} · Jianhu Feng¹ · Supei Zheng¹ · Xiaohan Cheng¹ · Yang Li²

Received: 16 May 2022 / Revised: 11 October 2022 / Accepted: 7 November 2022 /
Published online: 19 January 2023
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Abstract

This paper proposes a new version of the high-resolution entropy-consistent (EC-Limited) flux for hyperbolic conservation laws based on a new minmod-type slope limiter. Firstly, we identify the numerical entropy production, a third-order differential term deduced from the previous work of Ismail and Roe [11]. The corresponding dissipation term is added to the original Roe flux to achieve entropy consistency. The new, resultant entropy-consistent (EC) flux has a general and explicit analytical form without any corrective factor, making it easy to compute and a less-expensive method. The inequality constraints are imposed on the standard piece-wise quadratic reconstruction to enforce the pointwise values of bounded-type numerical solutions. We design the new minmod slope limiter as combining two separate limiters for left and right states. We propose the EC-Limited flux by adding this reconstruction data method to the primitive variables rather than to the conservative variables of the EC flux to preserve the equilibrium of the primitive variables. These resulting fluxes are easily applied to general hyperbolic conservation laws while having attractive features: entropy-stable, robust, and non-oscillatory. To illustrate the potential of these proposed fluxes, we show the applications to the Burgers equation and the Euler equations.

Keywords Hyperbolic conservation laws · Entropy production · Entropy-consistent (EC) flux · Slope limiter · High-resolution entropy-consistent (EC-Limited) flux

Mathematics Subject Classification 00A05

1 Introduction

Consider the hyperbolic system of conservation laws in one space dimension

✉ Supei Zheng
zsp2008@chd.edu.cn

Jianhu Feng
jhfeng@chd.edu.cn

¹ School of Sciences, Chang'an University, Xi'an 710064, Shaanxi, China

² Science and Technology on Combustion, Internal Flow and Thermostructure Laboratory, School of Astronautics, Northwestern Polytechnical University, Xi'an 710072, Shaanxi, China

$$\partial_t \mathbf{u} + \partial_x \mathbf{f}(\mathbf{u}) = 0, \quad (x, t) \in \mathbb{R} \times [0, +\infty); \quad (1)$$

here, $\mathbf{f}(\mathbf{u}): \mathbb{R}^N \rightarrow \mathbb{R}^N$ are smooth nonlinear flux functions of the conservative variables $\mathbf{u} \in \mathbb{R}^N$. Although initial conditions are smooth enough, solutions of (1) may develop singularities at a finite time, after which one must admit weak solutions. Yet, such weak solutions of the conservative equations are not unique. We would like to add the entropy-stable (ES) criteria to single out a unique physically relevant weak solutions.

Assume that system (1) is equipped with a given entropy pair (U, F) such that a strictly convex entropy function $U = U(\mathbf{u})$ augmented with an entropy flux $F = F(\mathbf{u})$ satisfies the following compatibility relation:

$$\mathbf{v}^T A = F_{\mathbf{u}}^T. \quad (2)$$

Here, $A = A(\mathbf{u})$ is the Jacobian matrix

$$A(\mathbf{u}) = \mathbf{f}_{\mathbf{u}}.$$

The entropy variables

$$\mathbf{v} = \mathbf{v}(\mathbf{u}) = U_{\mathbf{u}}.$$

Using the convexity of $U(\mathbf{u})$, the mapping $\mathbf{u} \rightarrow \mathbf{v}$ is one-to-one.

Consider the conservative semi-discrete form

$$\frac{d}{dt} \mathbf{u}_i(t) + \frac{1}{\Delta x} (\mathbf{f}_{i+1/2} - \mathbf{f}_{i-1/2}) = 0, \quad (3)$$

serving as the consistent approximation to the system conservation laws (1) with equidistant cells $I_i = [x_{i-1/2}, x_{i+1/2}]$. Here, $\mathbf{f}_{i+1/2} = \mathbf{f}(\mathbf{u}_i, \mathbf{u}_{i+1})$ is the Lipschitz continuous numerical flux which is consistent with the flux $\mathbf{f}(\mathbf{u})$ at the interface $x_{i+1/2}$ in the interval $[x_i, x_{i+1}]$, and $\Delta x = x_{i+1/2} - x_{i-1/2}$ is the spatial mesh size.

Multiplying by $\mathbf{v}^T$ and employing (3), a discrete cell entropy equality is sought

$$\frac{d}{dt} U_i(t) + \frac{1}{\Delta x} (F_{i+1/2} - F_{i-1/2}) = 0, \quad (4)$$

here, $F_{i+1/2}$ is the numerical entropy flux. We are interested for a uniquely physical solution which satisfies the following discrete cell entropy inequality (entropy stability):

$$\frac{d}{dt} U_i(t) + \frac{1}{\Delta x} (F_{i+1/2} - F_{i-1/2}) \leq 0. \quad (5)$$

In [10, 11], Ismail and Roe proposed a third-order differential terms of the *entropy production* to construct the entropy-consistent (EC) flux for Burgers equation and then heuristically extended it to the Euler equations. It is completely ES with a number of attractive features: it guarantees the amount of the entropy production is correct, exactly preserves the stationary contact discontinuity and does not capture the unphysical shock. However, the entropy dissipation should be neither too great nor too small. By adding the differential terms on the original Roe flux, the EC flux can more reasonably control the entropy dissipation across the discontinuity. In the present view, it is the most accurate flux for estimating entropy changes among ES fluxes. However, in those works, there is no direct analytical connection between the entropy production and the shock quality, and the EC dissipation term is determined only by a rough third-order differential term which is not

so satisfactory. It also seems quite hard to obtain the “reasonable” EC dissipation term for Euler equations. Furthermore, there is no more theory to apply the EC flux to other conservation laws.

In the present paper, we implement the second-order accurate entropy-conservative flux proposed by Tadmor [25] in the deduction of numerical entropy production. Then, we obtain the analytical expression of the numerical entropy production which is related to the shock quality, and quantify precisely how much the entropy dissipation is to be added. After that, by adding the numerical dissipation terms corresponding to the entropy production to the Roe flux, we construct the EC flux.

On the other hand, two numerical dissipation terms are added to the entropy-conservative one to construct the EC flux. One is an upwind Roe-type dissipation term to develop the Roe flux, and the other is an EC dissipation term proposed above to fix the entropy diffusion. Although the entropy-conservative flux is second-order accuracy, the EC flux is at most first-order accuracy due to the first-order accuracy dissipation terms. We aim at constructing numerical fluxes that have some good properties: (i) the fluxes have a second or higher order of accuracy in smooth parts of the solution; (ii) the fluxes produce numerical solution free from spurious oscillations; (iii) the fluxes produce high-resolution discontinuities, that is the number of mesh points in the transition zone is smaller than that of first-order fluxes. Fluxes of schemes satisfying these properties are called high-resolution fluxes as described by Harten [7].

In general, there are two main ways to develop the high-resolution flux. One is the flux limiter. This requires a high-order flux (greater than or equal to two) and a low-order flux (first-order). A flux limiter is a function which produces a convex combination of these fluxes. Sweby introduced the flux limiter to ensure the high resolution of numerical schemes [23], and then many scholars developed a series of excellent limiters [19, 20, 24]. In 2014, Liu and Feng proposed a compound limiter [17] which broadened the application field of flux limiters.

The other is the slope limiter. A slope limiter is a postprocessing tool which replaces the constant states (cell averages) by piece-wise linear or parabolic functions at first. The obtained values of the functions at the end points (left and right states for a cell) are usually called *boundary extrapolated values* by adding some inequality constraints on the functions; hence, the alternative name for the variable extrapolation method. Then, these values are used for low-order numerical fluxes and to obtain high-resolution high-order numerical fluxes. Note that the cell average in a cell is identical and thus the reconstruction process retains the conservation. Many slope limiters were proposed on finite volume [2, 5, 18, 27] and other methods [13, 14] to find high-order methods.

In view of the above, we aim at constructing a new minmod-type slope limiter based on the standard piece-wise quadratic reconstruction. We impose two inequality constraints to a two-point flux during the reconstruction to ensure the pointwise values of numerical solutions are bounded and separately select the proper slope limiter for each cell to obtain the left and right states. Unlike the previous works of taking a common slope limiter for a single cell, our new slope limiter pays more attention to the separate reconstruction of left and right states. The approach makes it possible to preserve more valid pointwise data and information. Moreover, the reconstruction technology is applied to the primitive variables instead of the conservative variables to maintain the equilibrium of the velocity and pressure across the interface.

The methodology that used in this paper includes the EC flux based on suitable dissipation terms and the slope limiter designed to improve the accuracy in a creative manner.

The paper is organized as follows. In Sect. 2, we obtain the entropy production by analyzing the behavior of a single cell and propose the EC flux with the reasonable corresponding dissipation terms. Section 3 presents a new slope limiter based on the piecewise quadratic reconstruction. Then, by combining the EC flux with the slope limiter, we develop the high-resolution entropy-consistent (EC-Limited) flux. A brief illustration for extending fluxes to two-dimensional (2-D) cases is done in Sect. 4. Several one-dimensional (1-D) and 2-D numerical experiments to the Burgers equation and Euler equations are presented in Sect. 5. Some conclusions and remarks are drawn in Sect. 6.

2 EC Flux

We discuss the conservation laws (1) with the entropy function U and introduce the entropy variable $\mathbf{v}$, so that

$$\partial_t U + \partial_x F = 0. \quad (6)$$

The system will be broken down as discontinuities develop. The entropy production in a domain Ω with boundary $\partial\Omega$ can be defined as

$$\dot{U} = \iint_{\partial\Omega} (\partial_t U + \partial_x F) dx dt = \oint_{\partial\Omega} (U dx + F dt). \quad (7)$$

Actually, it is very difficult to get the integral solved. In general, we analyze the process of the entropy production for two neighboring discrete states (x_i, x_{i+1}) with dual cell area (h_i, h_{i+1}) ($h_i = x_i - x_{i+1/2}$, $h_{i+1} = x_{i+1} - x_{i+1/2}$).

The conservation laws are discretized as

$$\partial_t \mathbf{u}_i = (\mathbf{f}_i - \mathbf{f}_{i+1/2})/h_i, \quad \partial_t \mathbf{u}_{i+1} = (\mathbf{f}_{i+1/2} - \mathbf{f}_{i+1})/h_{i+1}, \quad (8)$$

where $\mathbf{f}_{i+1/2}$ is the numerical flux of the semi-discrete scheme. Multiplying the corresponding $\mathbf{v}^T$ on the left

$$h_i \partial_t U_i = \mathbf{v}_i^T (\mathbf{f}_i - \mathbf{f}_{i+1/2}), \quad h_{i+1} \partial_t U_{i+1} = \mathbf{v}_{i+1}^T (\mathbf{f}_{i+1/2} - \mathbf{f}_{i+1}), \quad (9)$$

then the total cell entropy is given by

$$\partial_t (h_i U_i + h_{i+1} U_{i+1}) = -[\mathbf{v}^T \mathbf{f}]_{i+1/2} + [\mathbf{v}]_{i+1/2}^T \mathbf{f}_{i+1/2}, \quad (10)$$

where $[\cdot]_{i+1/2} = (\cdot)_{i+1} - (\cdot)_i$.

In addition, there is a relationship

$$\dot{U}_{i+1/2} = -[\mathbf{v}^T \mathbf{f}]_{i+1/2} + [\mathbf{v}]_{i+1/2}^T \mathbf{f}_{i+1/2} + [F]_{i+1/2} \quad (11)$$

between the entropy production and the numerical flux at interface $x_{i+1/2}$ in a finite volume method. If we choose $\mathbf{f}_{i+1/2}$ to make this expression vanish ($\dot{U}_{i+1/2} = 0$), it will result in the entropy-conservative flux $\mathbf{f}_{i+1/2}^C$, that is,

$$0 = -[\mathbf{v}^T \mathbf{f}]_{i+1/2} + [\mathbf{v}]_{i+1/2}^T \mathbf{f}_{i+1/2}^C + [F]_{i+1/2}. \quad (12)$$

Remark 1 Write the entropy-conservative flux from (12) as

$$[\mathbf{v}]_{i+1/2}^T \mathbf{f}^C = [\psi]_{i+1/2},$$

where $\psi = \mathbf{v}^T \mathbf{f} - F$ is the potential function. It is the well-known condition to construct the entropy-conservative flux (as seen in [26]).

There is another approach to obtain the entropy-conservative flux with the viscosity [25]

$$\mathbf{f}_{i+1/2}^C = \frac{1}{2}(\mathbf{f}_i + \mathbf{f}_{i+1}) - \frac{1}{2} \int_0^1 (2\xi - 1) B(\mathbf{v}(\xi)) d\xi [\mathbf{v}]_{i+1/2}. \quad (13)$$

It also satisfies the entropy-conservation condition (12)

$$0 = -[\mathbf{v}^T \mathbf{f}]_{i+1/2} + \frac{1}{2} [\mathbf{v}]_{i+1/2}^T \left(\mathbf{f}_i + \mathbf{f}_{i+1} - \int_0^1 (2\xi - 1) B(\mathbf{v}(\xi)) d\xi [\mathbf{v}]_{i+1/2} \right) + [F]_{i+1/2}. \quad (14)$$

Imposing the one-to-one mapping $\mathbf{u} \rightarrow \mathbf{v}$, the Jacobian

$$B(\mathbf{v}) = \frac{d\mathbf{f}}{d\mathbf{v}} = \frac{d\mathbf{f}}{d\mathbf{u}} \frac{d\mathbf{u}}{d\mathbf{v}}.$$

Here,

$$\mathbf{v}(\xi) = (1 - \xi)\mathbf{v}_i + \xi\mathbf{v}_{i+1}, \quad 0 \leq \xi \leq 1$$

is the simple straight-line segment path to connect the left state $\mathbf{v}_i$ and the right state $\mathbf{v}_{i+1}$.

According to [11], if $\mathbf{f}_{i+1/2} = \frac{1}{2}(\mathbf{f}_i + \mathbf{f}_{i+1})$,

$$\dot{U}_{i+1/2} = -[\mathbf{v}^T \mathbf{f}]_{i+1/2} + \frac{1}{2} [\mathbf{v}]_{i+1/2}^T (\mathbf{f}_i + \mathbf{f}_{i+1}) + [F]_{i+1/2}. \quad (15)$$

From (14) and (15), we can easily obtain the entropy production

$$\dot{U}_{i+1/2} = \frac{1}{2} [\mathbf{v}]_{i+1/2}^T \int_0^1 (2\xi - 1) B(\mathbf{v}(\xi)) d\xi [\mathbf{v}]_{i+1/2}. \quad (16)$$

We modify it as

$$\dot{U}_{i+1/2} = \frac{1}{2} [\mathbf{v}]_{i+1/2}^T \left| \int_0^1 (2\xi - 1) B(\mathbf{v}(\xi)) d\xi \right| [\mathbf{v}]_{i+1/2}, \quad (17)$$

where $\left| \int_0^1 (2\xi - 1) B(\mathbf{v}(\xi)) d\xi \right|$ just means that we should ensure the middle-integral matrix is the positive-definite matrix to ensure $\dot{U}_{i+1/2} > 0$. Then, we can obtain the dissipation term corresponding with the numerical entropy production

$$\mathbf{D}^{\text{EP}} = \frac{1}{2} \left| \int_0^1 (2\xi - 1) B(\mathbf{v}(\xi)) d\xi \right| [\mathbf{v}]_{i+1/2}. \quad (18)$$

This makes it easy for us to obtain this term in scalar cases.

In systems, there exists an invertible matrix R such that $A = \frac{d\mathbf{f}}{d\mathbf{u}} = R\Lambda R^{-1}$ and $\frac{d\mathbf{u}}{d\mathbf{v}} \approx R R^T$ showed by Barth in [1]. Then,

$$\mathbf{D}^{\text{EP}} = \frac{1}{2} \left| \int_0^1 (2\xi - 1) R(\mathbf{v}(\xi)) \Lambda(\mathbf{v}(\xi)) R^T(\mathbf{v}(\xi)) d\xi \right| [\mathbf{v}]_{i+1/2}, \quad (19)$$

where the diagnose matrix $\Lambda = \text{diag}(\lambda^1, \dots, \lambda^N)$ is the eigenvalue matrix of A , the matrix R is formed by the corresponding right eigenvector matrix, and R^T is the transpose of R .

We revise the above dissipation term as follows:

$$\mathbf{D}_{i+1/2}^{\text{EP}} = \frac{1}{2} R_{i+1/2} \left| \int_0^1 (2\xi - 1) \Lambda(\mathbf{v}(\xi)) d\xi \right| R_{i+1/2}^T [\mathbf{v}]_{i+1/2}, \quad (20)$$

where $R_{i+1/2}$ is an average matrix of R at the interface $x_{i+1/2}$. To avoid the analytical integration, which may be difficult, the integral above can be evaluated numerically using the Gauss-Legendre quadrature rule with enough accuracy. In this case, we obtain the new dissipative term

$$\mathbf{D}_{i+1/2}^{\text{EP}} = \frac{1}{2} R_{i+1/2} \left| \sum_{k=1}^q \Omega_k (2S_k - 1) \Lambda(\mathbf{v}(S_k)) \right| R_{i+1/2}^T [\mathbf{v}]_{i+1/2}, \quad (21)$$

where S_k are Gauss-points and Ω_k are associated weights of the Gaussian rule in the unit interval $[0, 1]$.

Usually, to simplify the calculation, we can use the three-point Gauss-Legendre quadrature rule to calculate the integral, with points S_k and weights Ω_k given by

$$S_{1,3} = \frac{1}{2} \pm \frac{\sqrt{15}}{10}, \quad S_2 = \frac{1}{2}, \quad \Omega_{1,3} = \frac{5}{9}, \quad \Omega_2 = \frac{8}{9}. \quad (22)$$

Roe added an upwind term (Roe-type dissipation term)

$$\mathbf{D}_{i+1/2}^{\text{ES}} = \frac{1}{2} R_{i+1/2} |\Lambda_{i+1/2}| R_{i+1/2}^T [\mathbf{v}]_{i+1/2} \quad (23)$$

to give the stability to the discretization and it leads to the ES flux

$$\begin{aligned} \mathbf{f}_{i+1/2}^{\text{ES}} &= \mathbf{f}_{i+1/2}^{\text{C}} - \mathbf{D}_{i+1/2}^{\text{ES}} \\ &= \mathbf{f}_{i+1/2}^{\text{C}} - \frac{1}{2} R_{i+1/2} |\Lambda_{i+1/2}| R_{i+1/2}^T [\mathbf{v}]_{i+1/2}. \end{aligned} \quad (24)$$

Here, $|\Lambda_{i+1/2}|$ is the matrix of absolute eigenvalues

$$|\Lambda_{i+1/2}| = \text{diag}(|\lambda_{i+1/2}^1|, \dots, |\lambda_{i+1/2}^N|). \quad (25)$$

The entropy across discontinuities increases, and the conservation mechanism breaks down simultaneously. The amount of the entropy increase is shown in the numerical flux mentioned above. Therefore, it is necessary to re-maintain the entropy conservation of the system by subtracting a term $\mathbf{D}_{i+1/2}^{\text{EP}}$. The new entropy-conservative flux is

$$\mathbf{f}_{i+1/2}^{\text{C-New}} = \mathbf{f}_{i+1/2}^{\text{C}} - \mathbf{D}_{i+1/2}^{\text{EP}}. \quad (26)$$

Then, by adding the Roe-type dissipative term $\mathbf{D}_{i+1/2}^{\text{ES}}$, we obtain the EC flux

$$\begin{aligned}
\mathbf{f}_{i+1/2}^{\text{EC}} &= \mathbf{f}_{i+1/2}^{\text{C-New}} - \mathbf{D}_{i+1/2}^{\text{ES}} \\
&= \mathbf{f}_{i+1/2}^{\text{C}} - \mathbf{D}_{i+1/2}^{\text{EP}} - \mathbf{D}_{i+1/2}^{\text{ES}} \\
&= \mathbf{f}_{i+1/2}^{\text{C}} - \frac{1}{2} R_{i+1/2} \left(|\Lambda_{i+1/2}| + \left| \sum_{k=1}^q \Omega_k (2S_k - 1) \Lambda(\mathbf{v}(S_k)) \right| \right) \mathbf{R}_{i+1/2}^T [\mathbf{v}]_{i+1/2}.
\end{aligned} \tag{27}$$

The ES flux or the EC flux developed above is only first-order accurate in space. To enhance the accuracy of the EC flux, a new slope limiter technique based on the idea of piece-wise quadratic data reconstruction is proposed. Other methods such as the ENO [8] or the WENO [16] interpolation also can be used to improve the accuracy of the EC flux.

3 EC-Limited Flux

In this section, we shall develop the EC-Limited flux based on a new slope limiter. We show the construction process of the slope limiter in scalar case, and then extend it to system. The new flux implies

- (i) the high-order accuracy obtained by data reconstruction;
- (ii) the reconstruction is constrained so as to avoid spurious oscillations.

Reviewing the standard piece-wise quadratic reconstruction at first. It is assumed that $r_i(x)$ is the reconstruction function of the smooth exact solution $u_i(x)$ which is k -th order accurate over the control volume $I_i = [x_{i-1/2}, x_{i+1/2}]$, equivalently, $|r_i(x) - u_i(x)| \leq O((\Delta x)^{k+1})$. The piecewise constant function cell average is

$$\bar{u}_i = \frac{1}{\Delta x} \int_{I_i} u_i(x) dx. \tag{28}$$

The local reconstruction satisfies

$$\frac{1}{\Delta x} \int_{I_i} r_i(x) dx = \frac{1}{\Delta x} \int_{I_i} u_i(x) dx = \bar{u}_i. \tag{29}$$

The Taylor expansion of r_i at the center x_i of the cell I_i is

$$r_i(x) = u_i(x_i) + \left. \frac{\partial u_i}{\partial x} \right|_{x_i} (x - x_i) + \left. \frac{\partial^2 u_i}{\partial x^2} \right|_{x_i} \frac{(x - x_i)^2}{2} + \dots, \tag{30}$$

and $u_i(x)$ also can be expanded in the same way to obtain

$$\begin{aligned}
\bar{u}_i &= \frac{1}{\Delta x} \int_{I_i} u_i(x) dx \\
&= \frac{1}{\Delta x} \int_{I_i} \left(u_i(x_i) + \left. \frac{\partial u_i}{\partial x} \right|_{x_i} (x - x_i) + \left. \frac{\partial^2 u_i}{\partial x^2} \right|_{x_i} \frac{(x - x_i)^2}{2} + \dots \right) dx \\
&= u_i(x_i) + 0 + \frac{1}{\Delta x} \cdot \int_{I_i} \left(\left. \frac{\partial^2 u_i}{\partial x^2} \right|_{x_i} \frac{(x - x_i)^2}{2} + \dots \right) dx.
\end{aligned} \tag{31}$$

Combining (30) with (31), we obtain

$$r_i(x) = \bar{u}_i + \frac{\partial u_i}{\partial x} \Big|_{x_i} (x - x_i) + \frac{1}{2} \frac{\partial^2 u_i}{\partial x^2} \Big|_{x_i} \left((x - x_i)^2 - \frac{1}{\Delta x} \int_{I_i} (x - x_i)^2 dx \right) + \dots \quad (32)$$

This can be explained as that the reconstruction solution at any point is the average over the control volume plus the high-order contribution of the reconstruction. In this paper, we choose the second-order reconstruction

$$r_i(x) = \bar{u}_i + \frac{\partial u_i}{\partial x} \Big|_{x_i} (x - x_i) + \frac{1}{2} \frac{\partial^2 u_i}{\partial x^2} \Big|_{x_i} \left((x - x_i)^2 - \frac{1}{\Delta x} \int_{I_i} (x - x_i)^2 dx \right) \quad (33)$$

to satisfy $|r_i(x) - u_i(x)| \leq O((\Delta x)^3)$. Then, placing $\bar{u}_i$ with u_i^n at the time t^n , we obtain

$$r_i(x) = u_i^n + H(x - x_i), \quad (34)$$

where

$$H(x - x_i) = \frac{\partial u_i}{\partial x} \Big|_{x_i} (x - x_i) + \frac{1}{2} \frac{\partial^2 u_i}{\partial x^2} \Big|_{x_i} \left((x - x_i)^2 - \frac{1}{\Delta x} \int_{I_i} (x - x_i)^2 dx \right). \quad (35)$$

In this paper, we only use the simplest central difference quotient to approximate the two partial derivatives. That is

$$\frac{\partial u_i}{\partial x} \Big|_{x_i} = \frac{u_{i+1}^n - u_{i-1}^n}{2\Delta x}, \quad \frac{\partial^2 u_i}{\partial x^2} \Big|_{x_i} = \frac{u_{i+1}^n - 2u_i^n + u_{i-1}^n}{\Delta x^2}. \quad (36)$$

Compared with the piece-wise linear reconstruction

$$H(x - x_i) = \frac{\partial u_i}{\partial x} \Big|_{x_i} (x - x_i), \quad (37)$$

the piece-wise quadratic reconstruction (34)–(35) adds a second-order space term, which makes the results more accurate and performs well in rarefaction waves and shock waves.

We propose the high-order accurate reconstruction

$$r_i^L = u_i^n + \varphi_L H(x_L - x_i), \quad r_i^R = u_i^n + \varphi_R H(x_R - x_i), \quad (38)$$

where $\varphi_L \in [0, 1]$ and $\varphi_R \in [0, 1]$ are slope limiter on the left and right. The values of $r_i(x)$ at extreme points $x_L = x_{i-1/2}$ and $x_R = x_{i+1/2}$ in cell I_i play the fundamental role. Different from the traditional one [2]

$$r_i(x) = u_i^n + \varphi_i H(x - x_i), \quad (39)$$

we pay more attention to separately construct slope limiters at extreme points which will be used for approximating the left and right states. The benefit is that we can use more data and information of the numerical solutions at the previous time layer.

The left and right states must satisfy

$$\begin{cases} \min(u_{i-1}^n, u_i^n) \leq r_i^L \leq \max(u_{i-1}^n, u_i^n), \\ \min(u_{i+1}^n, u_i^n) \leq r_i^R \leq \max(u_{i+1}^n, u_i^n) \end{cases} \quad (40)$$

to ensure that no new extreme values are generated locally. Then, by transforming the two formulas above, we obtain inequalities as follows:

$$\begin{cases} \min(u_{i-1}^n - u_i^n, 0) \leq r_i^L - u_i^n \leq \max(u_{i-1}^n - u_i^n, 0), \\ \min(u_{i+1}^n - u_i^n, 0) \leq r_i^R - u_i^n \leq \max(u_{i+1}^n - u_i^n, 0). \end{cases} \quad (41)$$

Next, we will take the first one as an example to illustrate the limiter we constructed.

When $u_{i-1}^n > u_i^n$, the inequality will be

$$0 < r_i^L - u_i^n < \max(u_{i-1}^n - u_i^n, 0), \quad (42)$$

that means

$$0 < \varphi_L H(x_L - x_i) < \max(u_{i-1}^n - u_i^n, 0). \quad (43)$$

Setting $\varphi_L \in [0, 1]$, we take the limiter on the left as

$$\varphi_L = \max\left(\min\left(1, \frac{u_{i-1}^n - u_i^n}{H(x_L - x_i)}\right), 0\right). \quad (44)$$

When $u_{i-1}^n < u_i^n$, the limiter is the same as above. In particular, when $u_{i-1}^n = u_i^n$, just take $\varphi_L = 0$ because that the solution is so smooth that the role of the limiter is not needed. It guarantees to meet the first inequality in (40).

Similar to the left, we get the limiter on the right

$$\varphi_R = \max\left(\min\left(1, \frac{u_{i+1}^n - u_i^n}{H(x_R - x_i)}\right), 0\right). \quad (45)$$

Usually, a very small constant $\varepsilon = 10^{-6}$ can be added to avoid the denominator being zero. We select two proper limiters in a single cell to satisfy the left and right data reconstruction conditions, respectively. It can automatically select the reconstructed data to maintain high-order accurate in smooth regions of the solution for selection of the original data to avoid spurious oscillations in discontinuous regions.

The EC flux (27) is

$$\mathbf{f}_{i+1/2}^{\text{EC}} = \mathbf{f}_{i+1/2}^{\text{EC}}(\mathbf{v}_i, \mathbf{v}_{i+1}), \quad (46)$$

and now, we want to replace the original values $\mathbf{v}_i$ and $\mathbf{v}_{i+1}$ with the reconstruction states $\mathbf{v}_i^R$ and $\mathbf{v}_{i+1}^L$ in the flux. This is our EC-Limited flux

$$\mathbf{f}_{i+1/2}^{\text{EC}} = \mathbf{f}_{i+1/2}^{\text{EC}}(\mathbf{v}_i^R, \mathbf{v}_{i+1}^L). \quad (47)$$

For scalar cases, this is very simple. But for systems, both the conservative variables $\mathbf{u}$ and the entropy variables $\mathbf{v}$ ($\mathbf{u}$ and $\mathbf{v}$ can convert to each other) are written in the primitive variables form. If we reconstruct $\mathbf{u}$ or $\mathbf{v}$ as a whole, the continuously changing equilibrium of each primitive variable may be broken. It is crucial to reconstruct the primitive variables at first. For instance, the entropy variable $\mathbf{v}$ for the 1-D Euler equations is

$$\mathbf{v} = \frac{\partial U}{\partial \mathbf{u}} = \left(\frac{\gamma - s}{\gamma - 1} - \frac{1}{2} \frac{\rho u^2}{p}, \frac{\rho u}{p}, -\frac{\rho}{p} \right)^T$$

(more details as seen in Sect. 5). It is obvious that the entropy variable $\mathbf{v} = \mathbf{v}(\rho, u, p)$ is a function of the primitive variables density ρ , velocity u , and pressure p . By reconstructing

the primitive variables, the new primitive variables are (ρ_i^R, u_i^R, p_i^R) , and then $\mathbf{v}_i^R$ for cell I_i at point $x_R = x_{i+1/2}$ is obtained.

4 Implementations in Multi-dimensions

For the 2-D conservation laws

$$\partial_t \mathbf{u} + \partial_x \mathbf{f}(\mathbf{u}) + \partial_y \mathbf{g}(\mathbf{u}) = 0, \quad (48)$$

where $\mathbf{f}$ and $\mathbf{g}$ are conservative fluxes in x and y directions, respectively. The semi-discrete scheme with cell $I_{ij} = [x_{i-1/2}, x_{i+1/2}] \times [y_{j-1/2}, y_{j+1/2}]$ is

$$\frac{d}{dt} \mathbf{u}_{ij} = -\frac{1}{\Delta x} (\mathbf{f}_{i+1/2,j} - \mathbf{f}_{i-1/2,j}) - \frac{1}{\Delta y} (\mathbf{g}_{i,j+1/2} - \mathbf{g}_{i,j-1/2}). \quad (49)$$

We can obtain the EC flux by extending the method dimension by dimension. It is as follows:

$$\begin{cases} \mathbf{f}_{i+1/2,j}^{\text{EC}} = \mathbf{f}_{i+1/2,j}^{\text{C}} - \frac{1}{2} R_{i+1/2,j}^x \mathbf{D}_{i+1/2}^{\text{EC}x} (R_{i+1/2,j}^x)^T [\mathbf{v}]_{i+1/2,j}, \\ \mathbf{g}_{i,j+1/2}^{\text{EC}} = \mathbf{g}_{i,j+1/2}^{\text{C}} - \frac{1}{2} R_{i,j+1/2}^y \mathbf{D}_{i,j+1/2}^{\text{EC}y} (R_{i,j+1/2}^y)^T [\mathbf{v}]_{i,j+1/2}. \end{cases} \quad (50)$$

Reconstructing the numerical flux in different directions, we can obtain the EC-Limited flux for the 2-D hyperbolic conservation laws.

5 Numerical Experiments

This section contains several 1-D and 2-D numerical experiments. All problems use uniform grids. The third-order strong stability preserving the Runge-Kutta method [6] is adopted in the time advancing. The solid line denotes the exact solution while the symbol lines represent the numerical solutions.

5.1 1-D Burgers Equation

In this subsection, we solve the Burgers equation

$$\partial_t u + \partial_x \left(\frac{1}{2} u^2 \right) = 0.$$

We know that the characteristic value $a = u$ is the wave speed and the entropy variable $v = u$. The entropy-conservative flux, ES flux, and EC flux for Burgers equation are

$$\begin{aligned} f_{i+1/2}^{\text{C}} &= \frac{1}{6} (u_i^2 + u_i u_{i+1} + u_{i+1}^2), \\ f_{i+1/2}^{\text{ES}} &= \frac{1}{6} (u_i^2 + u_i u_{i+1} + u_{i+1}^2) - \frac{1}{4} |u_i + u_{i+1}| (u_{i+1} - u_i), \\ f_{i+1/2}^{\text{EC}} &= \frac{1}{6} (u_i^2 + u_i u_{i+1} + u_{i+1}^2) - \frac{1}{4} |u_i + u_{i+1}| (u_{i+1} - u_i) - \frac{1}{12} |u_{i+1} - u_i| (u_{i+1} - u_i). \end{aligned}$$

Several experiments that we carried out are reported here with the added good properties of our new flux functions (EC and EC-Limited).

Example 1 Modelling rarefaction with a stationary shock

Consider the problem in $[-1, 1]$ with the initial condition

$$u(x, 0) = \begin{cases} -1, & \frac{1}{3} \leq |x| \leq 1, \\ 1, & |x| < \frac{1}{3}, \end{cases}$$

and compute until $t = 0.32$. This initial square wave will evolve into a rarefaction fan on the left and a stationary shock on the right. This problem contains a sonic point rarefaction. Numerical results in Fig. 1 confirm that the EC flux eliminates the spurious oscillations near the shock. This is one of the benefits of including more precise control of entropy at discontinuities. The EC-Limited flux precisely captures the whole solution with the slope limiter. Its results have no expansion shock, which is something that some methods failed to do.

Example 2 Modelling compression waves

Consider the following initial value problem:

$$u(x, 0) = \begin{cases} u_0, & |x| > 1, \\ u_0 - u_1 * \sin(\pi x), & |x| \leq 1 \end{cases}$$

in domain $[-1, 1]$, where the amplitudes $u_1 = 0.5$ and $u_0 = 0$. This is an initial value problem which consists of compression and expansion waves. Its normalized analytical solution will be in the form of

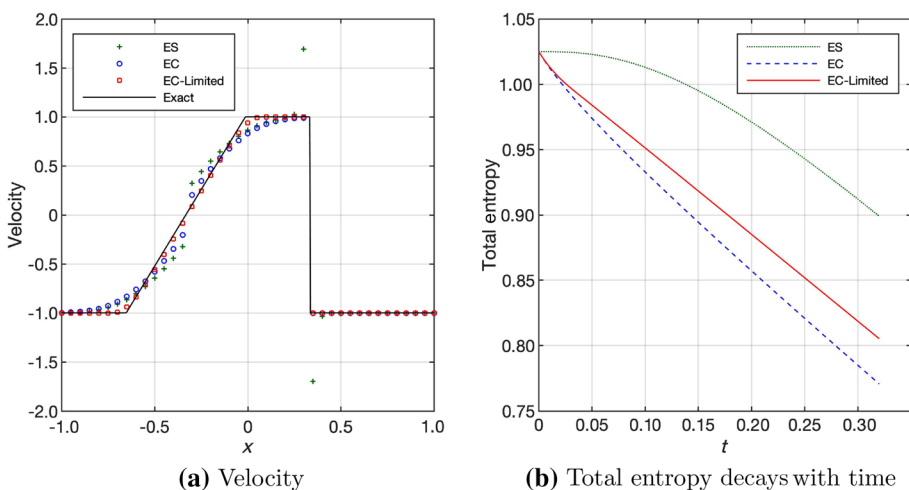
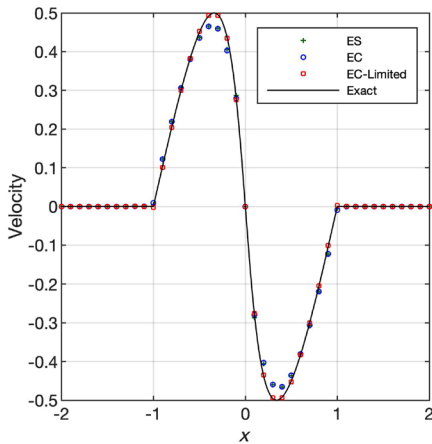
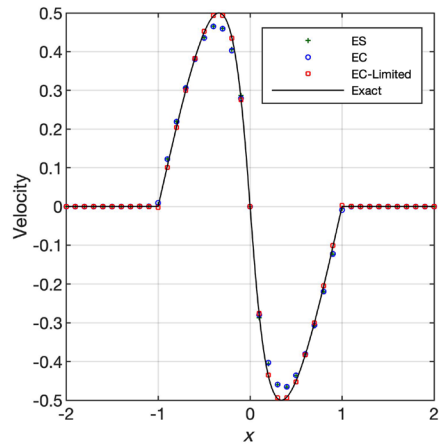


Fig. 1 Example 1, results of ES, EC, and EC-Limited on a mesh of 40 cells, **a** velocity and **b** total entropy decays with time

Table 1 Example 2, errors and orders of convergence of EC and EC-Limited in u at $t_1 = 0.32$

N	EC	Order	EC-Limited	Order
40	4.874 4E-3	—	3.118 3E-3	—
80	2.057 3E-3	1.24	7.559 3E-4	2.02
160	9.875 5E-4	1.06	1.736 4E-4	2.12
320	5.066 1E-4	0.96	3.167 8E-5	2.26

**(a)** Velocity at $t_1 = 0.32$ **(b)** Velocity at $t_1 = 0.96$ **Fig. 2** Example 2, results of ES, EC, and EC-Limited on a mesh of 40 cells, **a** velocity at $t_1 = 0.32$ and **b** velocity at $t_2 = 0.96$

$$u(x, t) = \frac{u_0}{u_1} - \sin\left(\pi x - \frac{u}{u_1} u_1 t\right).$$

We computed the solutions at $t_1 = 0.32$ and $t_2 = 0.96$. Table 1 lists the errors and the orders of convergence in u at $t_1 = 0.32$ obtained using our EC and EC-Limited fluxes, respectively. We see that the EC flux is first-order accuracy and the EC-Limited flux is second-order accuracy as we predicted. Figure 2 shows that our EC dissipative term is less effective when the solution is smooth that is consistent with the analysis. Our new slope limiter also presents good properties which can capture waves more accurately.

5.2 1-D Euler Equations

Consider the Euler equations of inviscid compressible gas dynamics

$$\frac{\partial}{\partial t} \begin{bmatrix} \rho \\ \rho u \\ E \end{bmatrix} + \frac{\partial}{\partial x} \begin{bmatrix} \rho u \\ \rho u^2 + p \\ (E + p)u \end{bmatrix} = \mathbf{0}.$$

Here, ρ , u , and p are the density, velocity, and pressure of the gas, respectively. The total energy is defined as $E = \frac{p}{\gamma-1} + \frac{\rho u^2}{2}$ where γ is the ideal gas constant. Defining the entropy function as $U = -\frac{\rho s}{\gamma-1}$ where $s = \ln p - \gamma \ln \rho$ is the physical entropy, we compute the entropy variable as

$$\mathbf{v} = \frac{\partial U}{\partial \mathbf{u}} = \left(\frac{\gamma - s}{\gamma - 1} - \frac{1}{2} \frac{\rho u^2}{p}, \frac{\rho u}{p}, -\frac{\rho}{p} \right)^T.$$

The ES flux and EC flux for Euler equations are

$$\begin{aligned} \mathbf{f}_{i+1/2}^{\text{ES}} &= \mathbf{f}_{i+1/2}^{\text{C}} - \frac{1}{2} R_{i+1/2} |\Lambda_{i+1/2}| S_{i+1/2} R_{i+1/2}^T [\mathbf{v}]_{i+1/2}, \\ \mathbf{f}_{i+1/2}^{\text{EC}} &= \mathbf{f}_{i+1/2}^{\text{C}} - \frac{1}{2} R_{i+1/2} \left(|\Lambda_{i+1/2}| + \left| \sum_{k=1}^q \Omega_k (2S_k - 1) \Lambda(\mathbf{v}(S_k)) \right| \right) S_{i+1/2} R_{i+1/2}^T [\mathbf{v}]_{i+1/2}, \end{aligned}$$

where $\mathbf{f}_{i+1/2}^{\text{C}}$ is the entropy-conservative flux, and $S_{i+1/2}$ is a matrix that produces the correct scaling. More details on this subject can be referred to [11], we will not go into details here to save space.

Example 3 Steady shock problem

The steady shock problem is a typical test case in many previous studies [11, 29]. Considering the left shock located at $x = 0$ with flow Mach number $M_\infty = 2$, the initial condition is given by Rankine-Hugoniot in the computational domain $[-1, 1]$ as follows:

$$(\rho, u, p) = \begin{cases} (\rho_l, u_l, p_l), & x < 0, \\ (\rho_r, u_r, p_r), & \text{otherwise,} \end{cases}$$

where

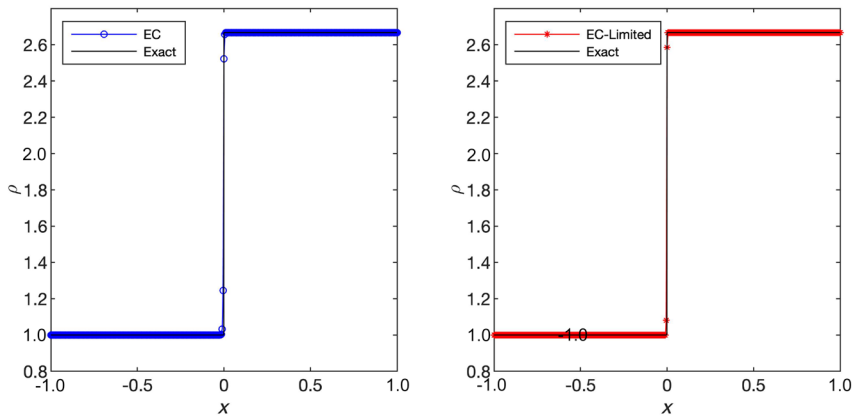
$$\begin{pmatrix} \rho_l \\ u_l \\ p_l \end{pmatrix} = \begin{pmatrix} \frac{1}{\gamma M_\infty^2} \\ 1 \\ 1 \end{pmatrix}, \quad \begin{pmatrix} \rho_r \\ u_r \\ p_r \end{pmatrix} = \begin{pmatrix} p_l^{\frac{2\gamma M_\infty^2 - (\gamma-1)}{\gamma+1}} \\ \frac{\gamma+1}{\gamma-1} \frac{p_l}{\rho_l} + 1 \\ \frac{\gamma+1}{\gamma-1} \frac{p_r}{p_l} \\ \sqrt{\gamma \frac{(2+(\gamma-1)M_\infty^2)p_r}{2\gamma M_\infty^2 + (1-\gamma)p_r}} \end{pmatrix}.$$

The initial condition is also the exact solution of the steady 1-D Euler equations. Figure 3a shows the numerical density distribution obtained by EC and EC-Limited fluxes along the computational domain. Obviously, both of all is essentially non-oscillatory. It is seen that the numerical shock of the EC flux has been smeared across a few grid points. However, the solution of the EC-Limited flux keeps the same shape as the exact solution and is quite sharp: just three cells across the shock. Figure 3c shows the evolution of the average residue which is defined as

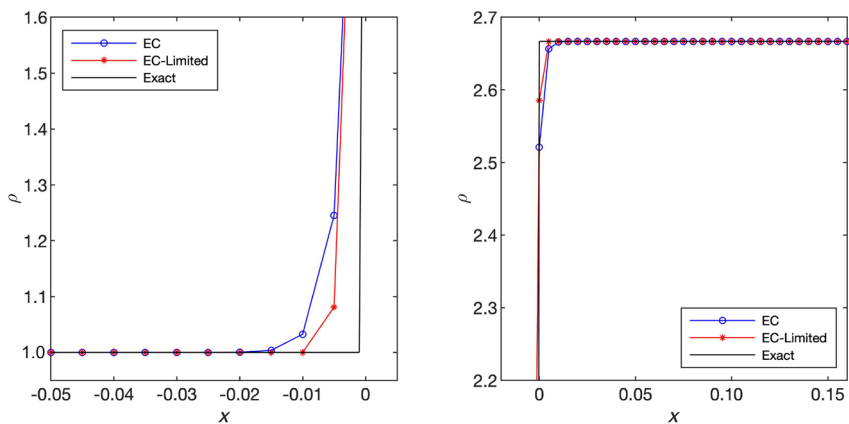
$$\text{Res}_A = \sum_{i=1}^N \frac{R_i}{N},$$

where R_i is the local residue defined as

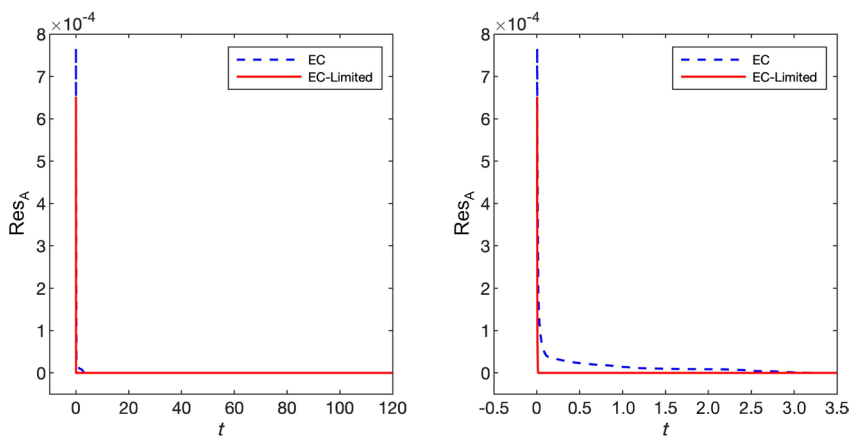
$$R_i = \frac{\partial \rho}{\partial t} \Big|_i = \frac{\rho_i^{n+1} - \rho_i^n}{\Delta t},$$



(a) Density distribution of EC (left) and EC-Limited (right)

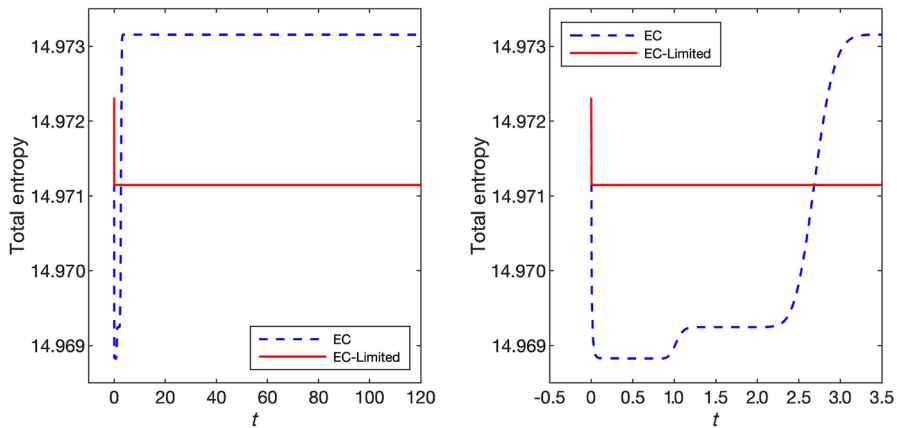


(b) Zoomed density distribution of EC and EC-Limited



(c) The average residue evolution of EC and EC-Limited

Fig. 3 Example 3, results of EC and EC-Limited on a mesh of 400 cells, **a** density, **b** zoomed density, **c** the revolution of the average residue, and **d** total entropy decays with time



(d) Total entropy decays with time

Fig. 3 (continued)

and N is the total number of grid points. Actually, compared with other methods, the average residue of two fluxes does not have periodical oscillations and converges to machine zero rapidly. We see that the indicator (the residue and total entropy) computed by the EC-Limited almost vertically drops and then be steady.

Example 4 Sod-shock tube problem

Consider the Sod problem with initial values

$$(\rho, u, p) = \begin{cases} (1, 0, 0.1), & x < 0.5, \\ (0.125, 0, 0.1), & \text{otherwise} \end{cases}$$

in the computational domain $[0, 1]$. Results at $t = 0.1644$ are shown in Fig. 4. The exact solution contains a left-going rarefaction wave, a middle contact discontinuity, and a right-going shock wave. Although the EC flux produces very similar results compared to those produced by the ES flux, its entropy dissipation is less than that of the ES flux. Obviously, the second-order EC-Limited flux captures the moving shock with only three intermediate cells.

Example 5 Low-density problem

Consider the low-density problem with the initial data

$$(\rho, u, p) = \begin{cases} (1, -2, 4), & x < 0.5, \\ (1, 2, 4), & \text{otherwise} \end{cases}$$

The results at $t = 0.1$ on the domain $[0, 1]$ are illustrated in Fig. 5. The solutions contain strong rarefaction waves on both sides and a contact discontinuous wave in the middle area. The characteristic of its solution is that the pressure is almost zero, because it is close to the vacuum in the middle part. Many numerical methods have negative pressure at this position. Also, we see that all fluxes mentioned in this paper can successfully solve this

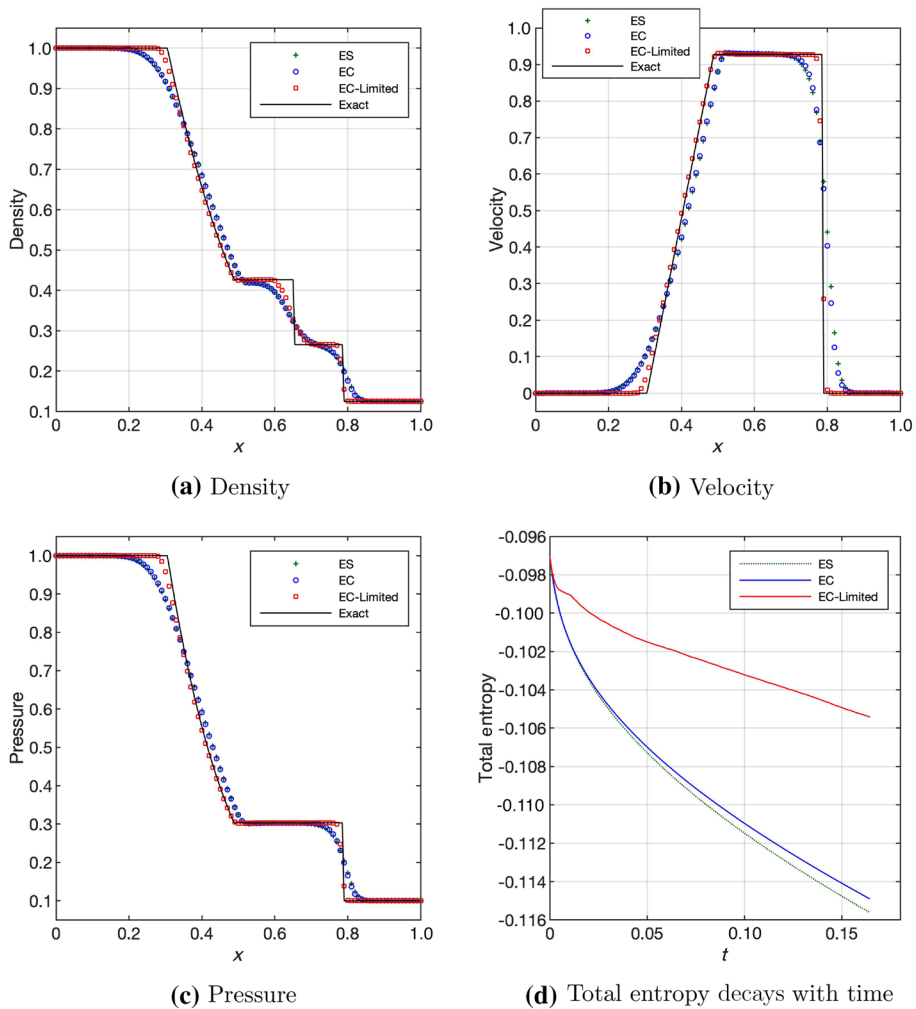


Fig. 4 Example 4, results of ES, EC, and EC-Limited on a mesh of 100 cells, **a** density, **b** velocity, **c** pressure, and **d** total entropy decays with time

problem. There is a critical flattening in the results of the EC-Limited flux in pressure, which is consistent with the exact solution.

Example 6 Two interacting blast waves problem

This problem is often used to examine the ability of numerical fluxes to capture shocks [9, 12]. The initial value condition is

$$(\rho, u, p) = \begin{cases} (1, 0, 1000), & -0.5 < x < -0.4, \\ (1, 0, 0.01), & -0.4 < x < 0.4, \\ (1, 0, 100), & 0.4 < x < 0.5. \end{cases}$$

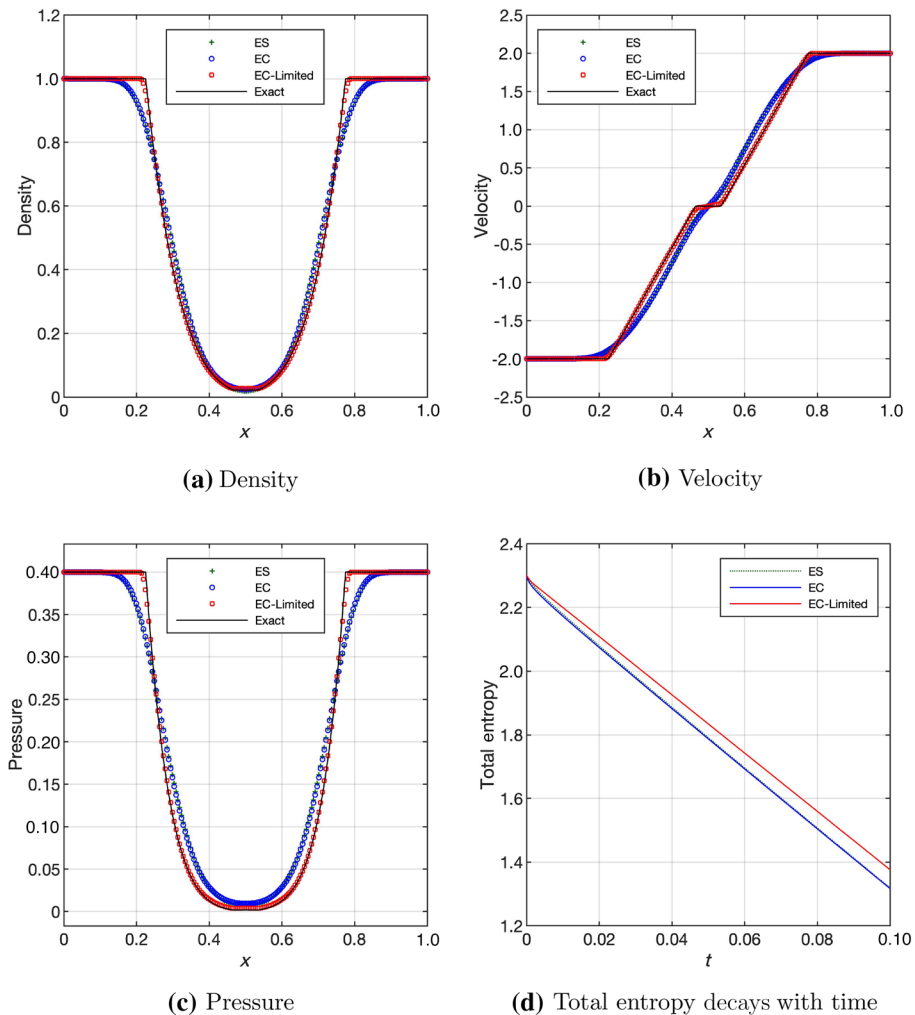


Fig. 5 Example 5, results of ES, EC, and EC-Limited on a mesh of 100 cells, **a** density and **b** velocity, **c** pressure, and **d** total entropy decays with time

The computing time is $t = 0.038$ on the domain $[-0.5, 0.5]$. The numerical density computed by the EC-Limited flux in [19] with 4 000 cells is taken as the “exact” solution which is denoted by the solid line. The numerical densities comparisons computed by the EC flux and EC-Limited flux are shown in Figs. 6a and 6b, respectively. We can easily find that the EC flux and EC-Limited flux are also suitable for complex problems. In addition, the result of the EC-limited still shows good characters of the new slope limiter proposed in Sect. 4.

Example 7 Shock entropy wave interaction problem

The solution of the shock entropy wave interaction problem contains both shocks and complex smooth region structures [22]. This problem is a typical example that is often

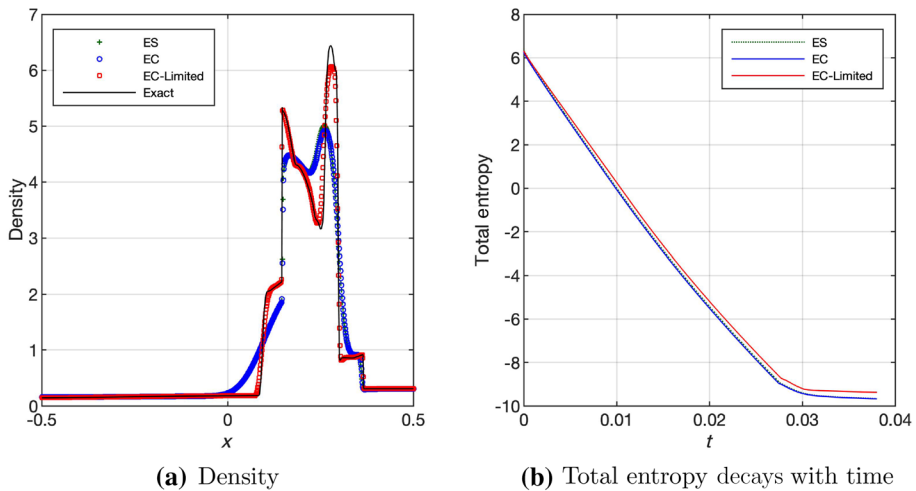


Fig. 6 Example 6, results of ES, EC, and EC-Limited on a mesh of 800 cells, **a** density and **b** total entropy decays with time

used to show the advantage of a high-resolution method. In this paper, we use the following initial condition:

$$(\rho, u, p) = \begin{cases} (3.857143, 2.629369, 10.333333), & -5 < x < -4, \\ (1 + 0.2 \sin(5x), 0, 1), & -4 \leq x < 5, \end{cases}$$

which describes a moving shock interacting with sine waves in density. The computing time is $t = 1.8$ on the domain $[-5, 5]$. Also, the result of the EC-Limited fluxes in [19] with 4 000 cells is taken as the “exact” solution. The detailed profiles of the density and velocity are exhibited in Figs. 7a and 7b, in which we can see the solution of the EC-Limited flux is still far superior to the EC flux. This case demonstrates that the new slope limiters will show a better resolution for the solutions which contain complicated smooth region structures.

5.3 2-D Burgers Equation

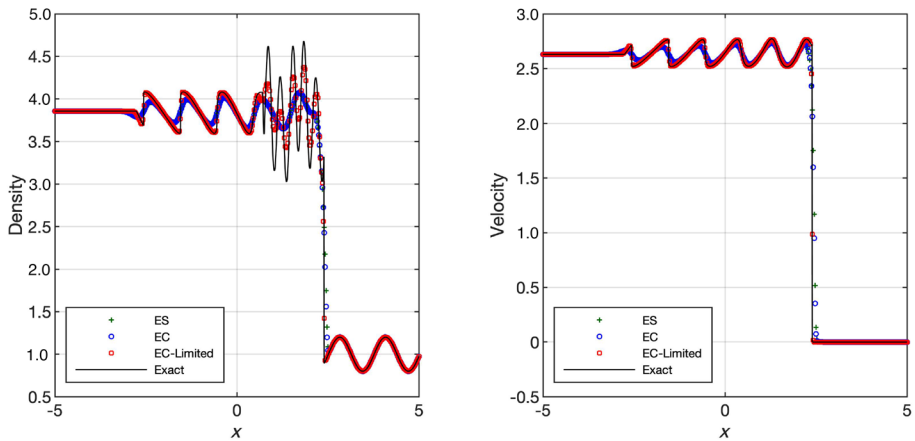
We solve the 2-D Burgers equations

$$\partial_t u + \partial_x f(u) + \partial_y g(u) = 0,$$

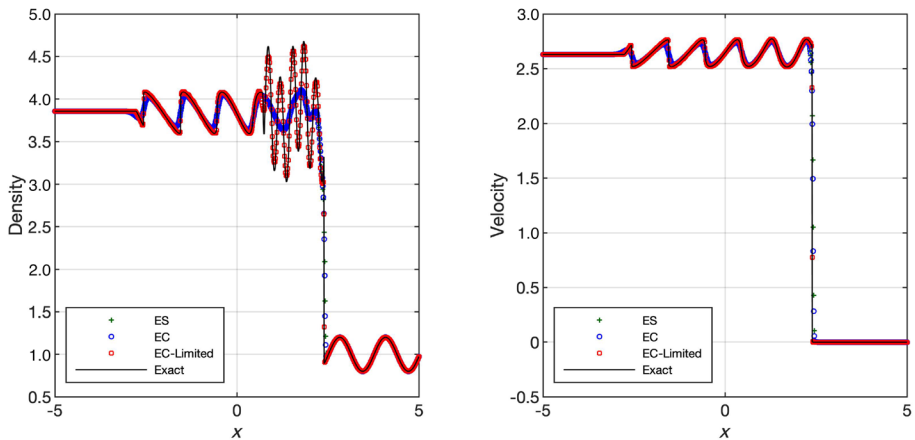
where $f = g = u^2/2$ and the other variables have the same meaning as in the 1-D case.

Example 8 Discontinuous initial value problem

We applied two new EC and EC-Limited fluxes on 2-D Burgers equations with the initial state



(a) Results of EC and EC-Limited on grid of 400 cells

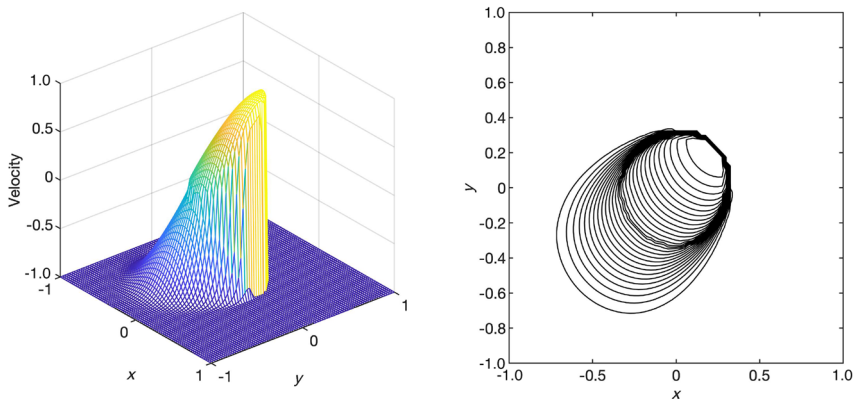
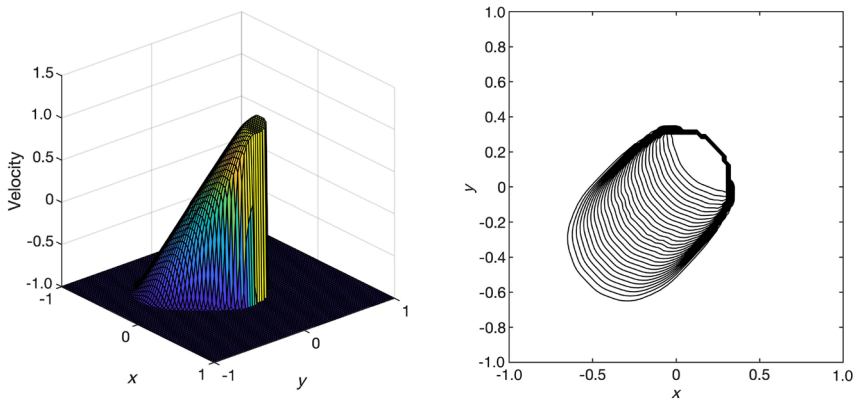
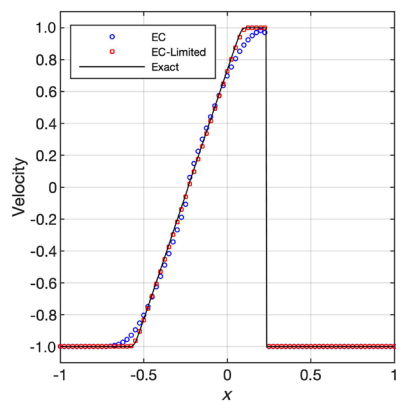
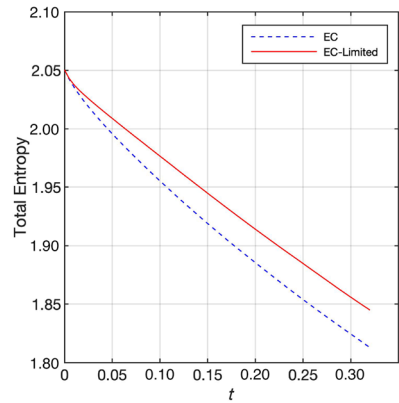


(b) Results of EC and EC-Limited on a grid of 800 cells

Fig. 7 Example 7, density and velocity distribution of ES, EC, and EC-Limited on two grids: 400 cells **a** and 800 cells **b**

$$u(x, y, 0) = \begin{cases} 1, & x^2 + y^2 < (1/3)^2, \\ -1, & \text{otherwise.} \end{cases}$$

The surface and contour plots of the velocity at time $t = 0.32$ computed by EC and EC-Limited on the domain $[-1, 1] \times [-1, 1]$ are shown in Figs. 8a and 8b. Figures 8c and 8d show the radial cut plot at $x = y$ and the total entropy decay with time. It shows that the solution of the EC-Limited is the best, whose surface plots are flatter in contact discontinuity waves. This experiment preliminarily tells us that the slope limiters we proposed are also effective in the 2-D case.

(a) Results of EC on a grid of 80×80 cells(b) Results of EC-Limited on a grid of 80×80 cells(c) Radial cut plot at $x = y$ 

(d) Total entropy decays with time

Fig. 8 Example 8, results on a grid of 80×80 cells: the velocity surface plots and contour plots of **a** EC, **b** EC-Limited, **c** radial cut plot at $x = y$, and **d** total entropy decays with time

5.4 2-D Euler Equations

Solving the nonlinear system of 2-D Euler equations

$$\frac{\partial}{\partial t} \begin{bmatrix} \rho \\ \rho u \\ \rho v \\ E \end{bmatrix} + \frac{\partial}{\partial x} \begin{bmatrix} \rho u \\ \rho u^2 + p \\ \rho uv \\ (E + p)v \end{bmatrix} + \frac{\partial}{\partial y} \begin{bmatrix} \rho v \\ \rho uv \\ \rho v^2 + p \\ (E + p)v \end{bmatrix} = \mathbf{0},$$

where u and v are velocities in x and y directions, respectively, and

$$E = \frac{p}{\gamma - 1} + \frac{\rho(u^2 + v^2)}{2}.$$

Two 2-D Riemann problems for gas dynamics-configuration (refer [15] for details) are considered in the following. Different from the 1-D Euler equations, this 2-D case consists of four wave patterns including the rarefaction $(\bar{R}, \bar{R})$, shock $(\bar{S}, \bar{S})$, and contact-wave (J^+, J^-) , which pose a computational challenge for high-resolution fluxes.

Example 9 Riemann problem for Configuration 3

The initial data is

$$(\rho, u, v, p)(x, y, 0) = \begin{cases} (1.5000, 0, 0, 1.5000), & x > 0.5, \quad y > 0.5, \\ (0.5323, 0, 0, 0.3000), & x < 0.5, \quad y > 0.5, \\ (0.1380, 1.2060, 1.2060, 0.0290), & x < 0.5, \quad y < 0.5, \\ (0.5323, 0, 1.2060, 0.3000), & x > 0.5, \quad y < 0.5. \end{cases}$$

The solution structure of this configuration is

$$\begin{array}{ccc} & \bar{S}_{21} & \\ \bar{S}_{32} & & \bar{S}_{41} \\ & \bar{S}_{34} & \end{array}$$

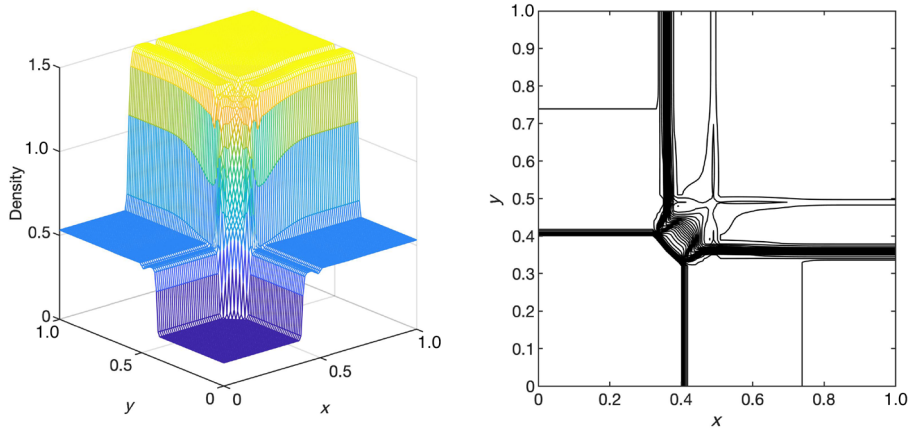
The surface and contour plots of density at time $t = 0.3$ computed by the EC flux and EC-Limited flux on the domain $[0, 1] \times [0, 1]$ are shown in Fig. 9. It is seen that the surface plots of the EC-Limited flux at four shock waves are very sharp, and the corresponding contour plots are concentrated. This test case, once again, demonstrates the advantage of our slope limiter for 2-D calculations.

Example 10 Riemann problem for Configuration 16

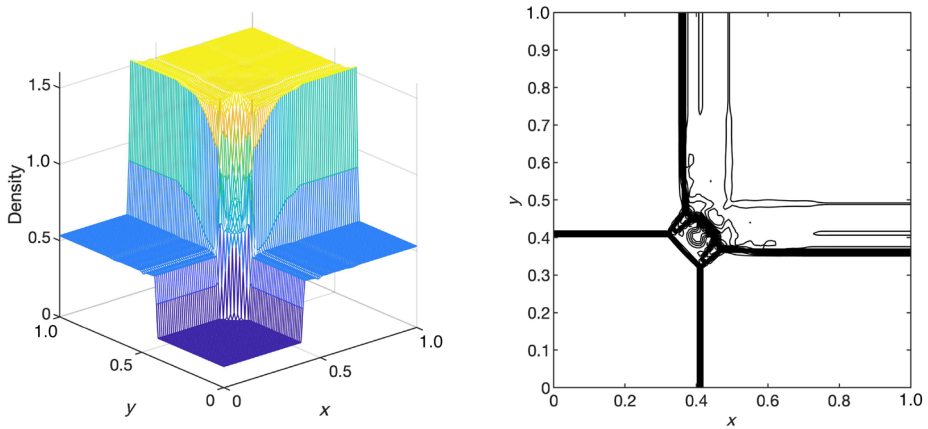
The initial data is

$$(\rho, u, v, p)(x, y, 0) = \begin{cases} (0.5313, 0.1000, 0.1000, 0.4000), & x > 0.5, \quad y > 0.5, \\ (1.0222, -0.6179, 0.1000, 1.0000), & x < 0.5, \quad y > 0.5, \\ (0.8000, 0.1000, 0.1000, 1.0000), & x < 0.5, \quad y < 0.5, \\ (1.0000, 0.1000, 0.8276, 1.0000), & x > 0.5, \quad y < 0.5. \end{cases}$$

The solution structure of this configuration is



(a) Results of EC on a grid of 100×100 cells



(b) Results of EC-Limited on a grid of 100×100 cells

Fig. 9 Example 9, results on a grid of 100×100 cells: the velocity surface plots and contour plots of **a** EC and **b** EC-Limited

$$\begin{array}{c} \bar{R}_{21} \\ \bar{S}_{32} \quad \bar{S}_{41} \\ J_{34}^+ \end{array}$$

The surface and contour plots of density at time $t = 0.2$ computed by the EC flux and EC-Limited flux on the domain $[0, 1] \times [0, 1]$ are shown in Fig. 10. The results depict that our EC flux solves this complex problem well and the EC-Limited flux can achieve high resolution.

Example 11 Rayleigh-Taylor instability

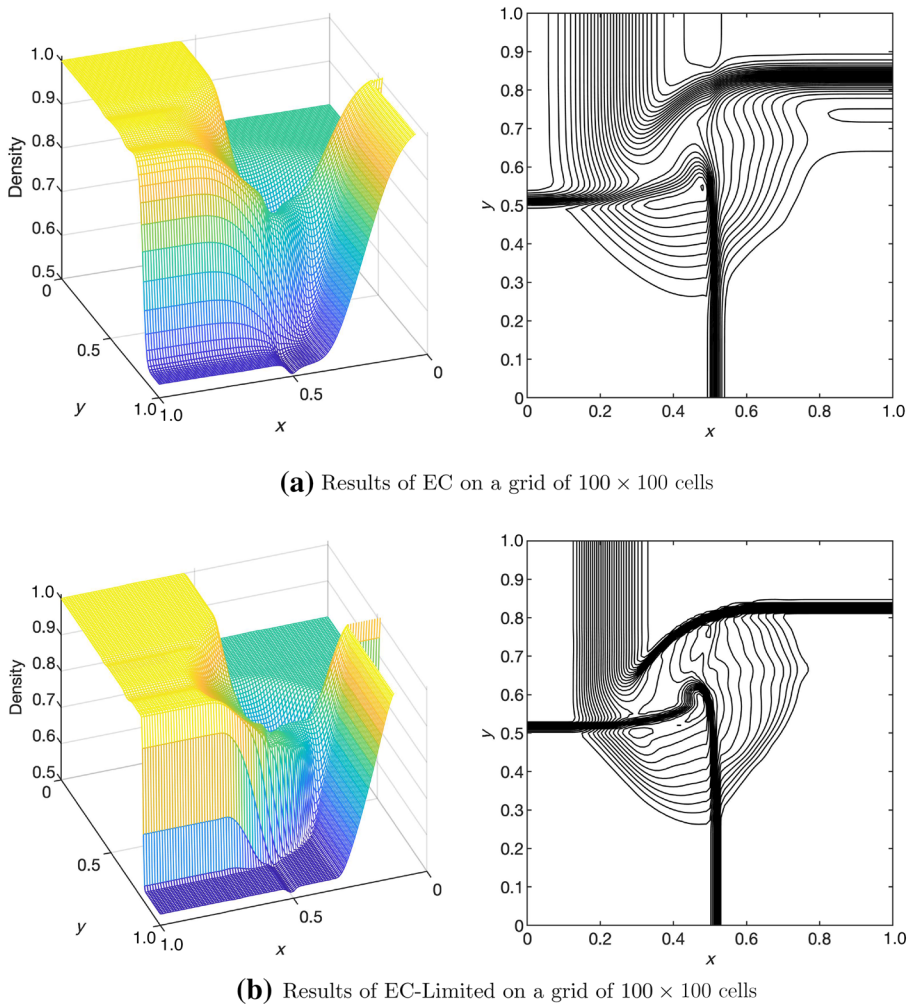


Fig. 10 Example 10, results on a grid of 100×100 cells: the velocity surface plots and contour plots of **a** EC and **b** EC-Limited

The Rayleigh-Taylor instability problem can be used as an effective test example to test the resolution of a numerical method. Here we examine the interaction between two fluids using the EC and EC-Limited fluxes, with the same initial conditions as those used in [21, 28]. The computational domain is a $[0, 0.25] \times [0, 1]$ rectangle. It happens on an interface between fluids with different densities when an acceleration is directed from the heavy fluid (fluid with density = 2) on the bottom to the light fluid (fluid with density = 1) on the top. The schlieren images and density contours with 15 equally spaced contour lines from $\rho = 0.952\,269$ to $\rho = 2.145\,89$ at time $t = 1.95$ with two different uniform meshes 60×240 and 120×480 are shown in Fig. 11. The heavy fluid penetrates into the top part of the domain, causing the light fluid to descend vertically along the two sides and produce Taylor-bubble. A classic mushroom structure emerges in the process. It can be seen that

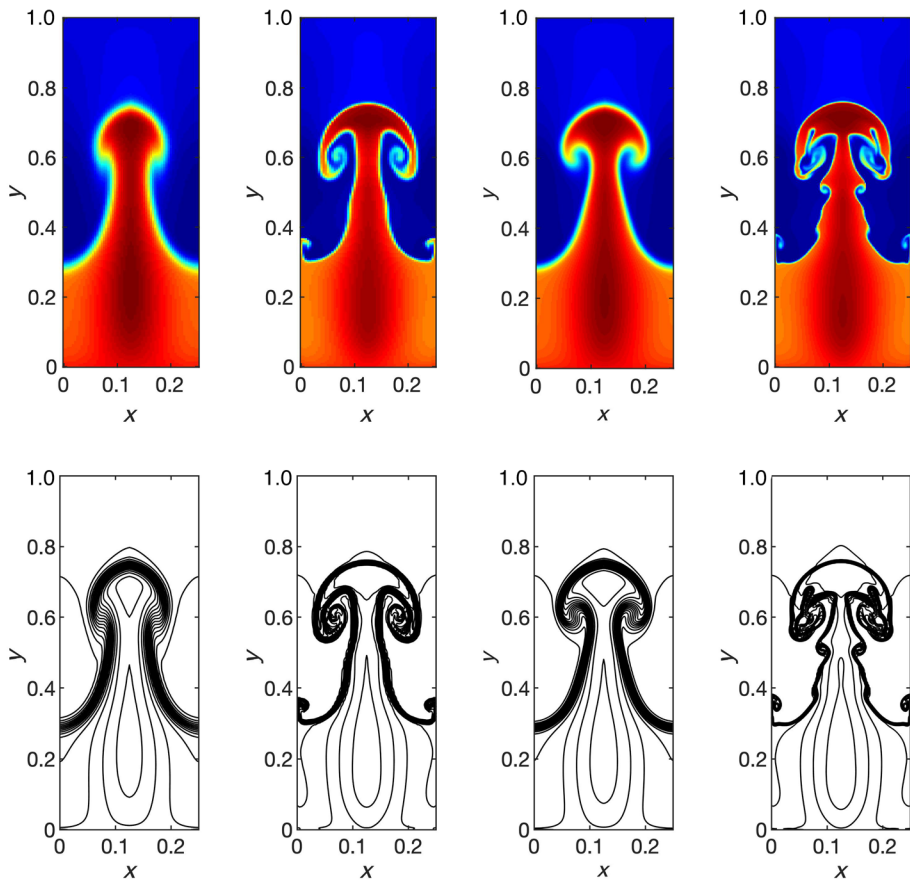
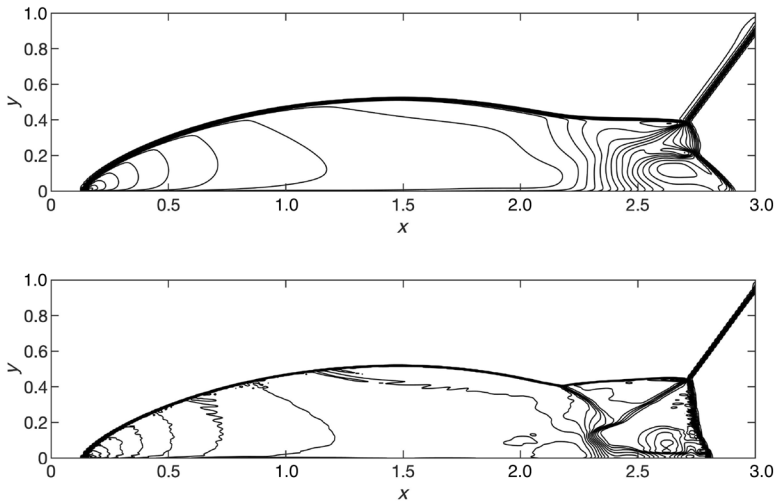


Fig. 11 Example 11, the schlieren images (top) and contour plots (bottom) of density, from left to right: EC and EC-Limited with 60×240 cells, EC and EC-Limited with 120×480 cells

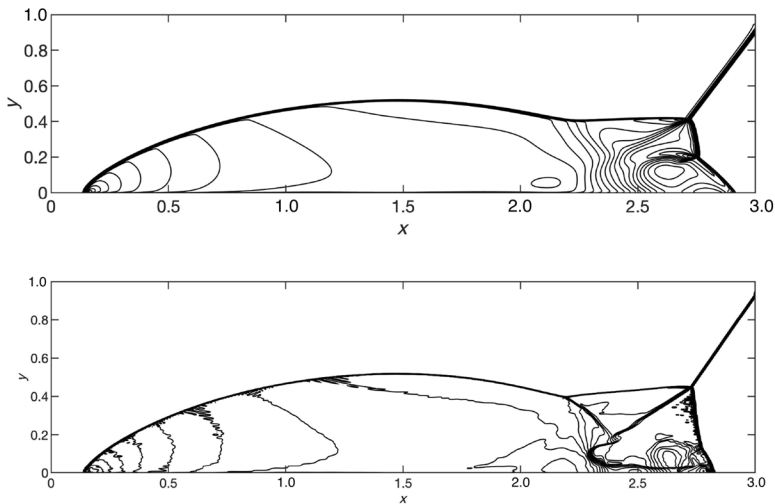
there is good agreement in the interfacial contours between the simulations and experiments [28], with EC-Limited limiting the interface to a thin layer.

Example 12 Double Mach reflection problem

This problem was initially proposed in detail by Woodward and Colella [27], and has been extensively studied in literature [3, 4, 21]. It is a good test example for high-resolution schemes. The computational domain is $[0, 4] \times [0, 1]$, and the left shock flow with Mach number $M_\infty = 10$ is positioned at $x = \frac{1}{6}$, $y = 0$ with a 60° angle with the x -axis. The right-moving shock impact the reflecting wall lies at bottom with domain $x = [\frac{1}{6}, 4]$ with forming the double Mach reflection. In Fig. 12, we display the density contours (30 equally spaced contour lines from $\rho = 2$ to $\rho = 20$) with two mesh of cells 240×240 and 480×480 on the domain $[0, 3] \times [0, 1]$. It is clear that a significant resolution difference between the EC and the EC-Limited. It is shown that the “blown-up” region around the double Mach stems in



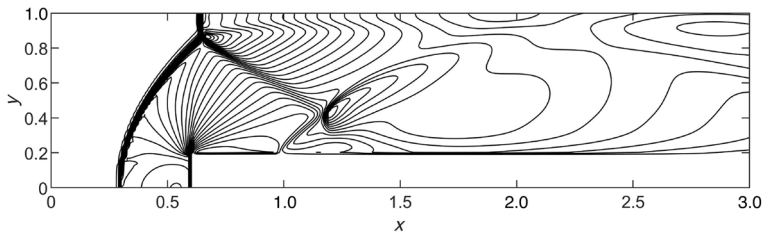
(a) Results of EC and EC-Limited on a grid of 240×240 cells



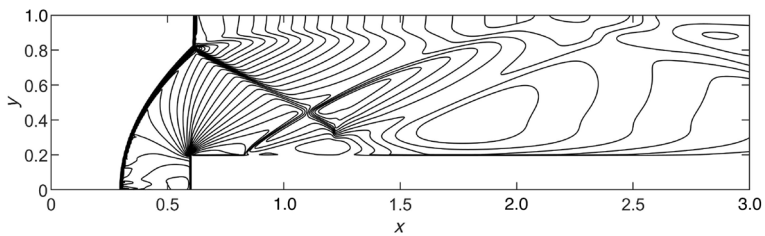
(b) Results of EC and EC-Limited on a grid of 480×480 cells

Fig. 12 Example 12, contour plots of density: **a** EC (up) and EC-Limited (bottom) with 240×240 cells and **b** EC (up) and EC-Limited (bottom) with 480×480 cells

the lower left corner of the result of the EC is abnormal. It means that the dissipation of EC is not enough and the EC-Limited gives a much better resolution for the complicated flow structure in this portion. We can see that the vortex in “blown-up” region of the EC-Limited has taken shape to the eye. It is no doubt that our slope limiter can modify the EC flux properly.



(a) Results of EC and EC-Limited on a grid of 240×80 cells



(b) Results of EC and EC-Limited on a grid of 480×160 cells

Fig. 13 Example 13, contour plots of density: **a** EC (up) and EC-Limited (bottom) with 240×80 cells and **b** EC (up) and EC-Limited (bottom) with 480×160 cells

Example 13 Forward facing step problem

This problem is also a typical test for the high-accurate method [4, 27]. A left uniform Mach 3 flow into a wind tunnel $[0, 3] \times [0, 1]$ containing a step located in $[0.6, 3] \times [0, 0.2]$. The time evolution, up to time 4, of the density distribution in the wind tunnel is displayed in Fig. 13, using two very uniform grid of 240×80 and 480×160 . The shock flow interact with the solid step at first, and then arrive at the upper boundary layer with forming the

second reflection. It clearly sees that the improve resolution, especially at the upper slip line from the triple point.

6 Conclusions

The EC flux is constructed by determining the dissipation term which can produce enough entropy dissipations for the entropy-conservative one. Different from Ismail and Roe's EC flux, it is a more widely applicable method without any additional artificial parameter adjusting. We then develop the EC-Limited flux with a new type of slope limiter which is based only on the characteristics of the reconstructed data. In the process, we creatively propose separately reconstructing the left and right states using the primitive variables instead of the conservative variables. Good properties and the validity of the slope limiter have been proved by many tests. The two new ideas in the construction of the slope limiter can still be used in the applications for multi-dimensional conservation laws on unstructured grids. These two fluxes we proposed in this work are very suitable for combining with other methods such as the discontinuous Galerkin methods. Some appropriate corrections such as smoothing the directional derivatives in data reconstruction may need to be done in future work.

Acknowledgements This work is supported by the National Natural Science Found Project of China through project number 11971075.

Compliance with Ethical Standards

Conflict of Interest The authors declare that they have no conflict of interest.

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